Just a cup...

JOHN GOUGH uses a range of simple materials to facilitate students' understandings of measurement concepts.

In several earlier articles (Gough 1999 to 2007) I have discussed some of the conceptual complexities, and conceptually challenging experiences, related to length, area, surface area, volume, mass, halving, and similar issues. I have argued that non-standard examples, and the non-connection between some of these concepts, need careful, open-ended exploration. The following discussion that extends these ideas begins in the kitchen.

Take one cup of… flour? sugar? milk? oil? rice? Recipes often start this way — pancakes, scones, cakes, biscuits, muffins, pasta, risotto, soups, and so on. The metric cup commonly used in Australia is 250 millilitres (ml) but other volumes are designated cups in other contexts. For the purposes of this article a cup is defined as 200 ml, or 200 cubic centimetres (a cubic centimetre is 1 centimetre × 1 centimetre × 1 centimetre, or 1 cm^3). The activities described are readily adaptable to other volumes. The activities presented are essentially about looking at different configurations of a given volume rather than about cups.

As soon as we start assembling a cup of this, and a cup of that, we find that this convenient kitchen volume of material can start to look very different: not all measuring cups look the same! What is in the cupful now may look different when it is tipped out! The Swiss psychologist Jean Piaget emphasised the crucial role of "conservation" when

young students begin working with measurement and counting (e.g., Bobis, Mulligan & Lowrie, 2004, pp. 7, 194, 203).

A child who is unable to "conserve" will believe that when a ball of playdough is squashed flat, or rolled out in a long thin snake, or broken into many small balls, there is more playdough. The child who has achieved conservation will not be confused by squashing playdough (or distorting the original quantity in some other way). This child will realise that squashing, and similar distortions, may change the shape (or arrangement) but will not alter the amount of material. Moreover, the child will understand that any distortion can be undone, rolled back into one ball, or otherwise reversed, so that the initial quantity, along with its original shape, can be re-obtained. This is the Piagetian principle of reversibility.

Later, even when students have developed robust reversibility, they are still prone to believing that if two objects have the same volume, they will have the same surface area, and the total length of their edges (a counterpart to two-dimensional perimeter) will be the same (e.g., Gough 1999; 2001; 2003; 2004; 2007a; 2007c). The following activities will challenge these, and similar misconceptions.

Our 200 ml measuring cup suggests some interesting mathematical activities, along with some serious challenges to our established ability to conserve volume (and length, area, surface area, and mass). The following activities may be attempted using connectable centimetre-cubes, or MAB minis and longs, or flour, rice, sand, plasticine, clay, or playdough.

An extension to these activities can include a visit to a timber or lumber yard. Softwood, balsa wood, hardwood, plywood and chipboard come in different thicknesses, sometimes less than 1 cm. Dressed timbers come in different square or (non-square) rectangular cross-sections, and dowels and quads come in different sizes, and with different cross-sections. Alternatively, consider using thick cardboard, cut in congruent pieces, and assembled in layers, like plywood.

Digital photos of the actual materials, in different configurations, are an excellent aid to memory. They are also an attractive way of displaying some of the more unusual results of these activities. Why not make a class gallery, a class book, or a digital slide-show of pictures of some interesting alternative "cupfuls"?

For each activity, consider which parts of the discussion might be:

- addressed to the teacher; and
- used directly with students.

Also, for each activity, two generic questions apply:

- What do you notice?
- Can you explain why?

Activity 1 — Spreading sand: Volume turns into area?

Take a 200 ml cup of sand or dough.
(Note: if you feel it is wrong, or seems strange, to measure dough using the fluid-based unit of millilitres, simply refer to 200 ml of dough as 200 cubic-centimetres of dough.)

Tip it into a dinner plate, and spread the material as evenly as possible. Look at the size of the area of material. How thick or "deep" is it on the plate?

Do the same with rectangular baking pans of different sizes.

Similarly, use cylindrical bottles, jugs, saucepans, and vases.

Use a funnel (hold it steady) to trickle 200 ml of fine sand onto a wide plate. Consider the cone-like result. A spatially and visually compact cupful may look very different when spread.

Consider the information given on a tin of house paint, suggesting that 1 litre of paint will cover 14 to 16 square metres of wall: how thin is the paint on that wall? Is it thicker or thinner than a human hair?

Activity 2 — Rectangular bricks: Constant volume and shape-type, different measurements

Take 200 ml of centicubes.
(Note: if you feel it is wrong, or seems strange, to specify a quantity of centicubes using the fluid-based unit of millilitres, simply refer to so many cubic centimetres of centicubes. In practice, teachers should help students see that millilitres and cubic centimetres are interchangeable.)

How many different ways can these be used to make rectangular "bricks" with a total volume of 200 cubic centimetres? Here are some examples. Start with four centicube "sticks" of 1 cm × 1 cm × 50 cm. These four sticks can be joined end-to-end, to make a 1 cm × 1 cm × 200 cm "pole". Does the "pole" look much like the "cup"? Imagine the pole as a rather long stirring rod, or swizzle stick, used to mix a drink in a 200 ml jug. Better still, actually make the "pole" and place it alongside a 200 ml block: it makes you think!

Alternatively, the four 50 cm "sticks" can make a one-centimetre-thick "slab" that has a total volume of 50 cm × 4 cm × 1 cm. What other one-centimetre-thick rectangular slabs can you make with different length and width?

In a timber-yard you can buy dressed square cross-section beading (or some similar technical term) with cross-section 9 mm × 9 mm. Two metres of this has a total volume of less than 200 ml — how do you know? How long would a piece of 9 mm × 9 mm beading have to be to be exactly 200 cubic centimetres in volume?

Consider this thicker and wider 2 cm × 4 cm × 25 cm slab (Figure 2).

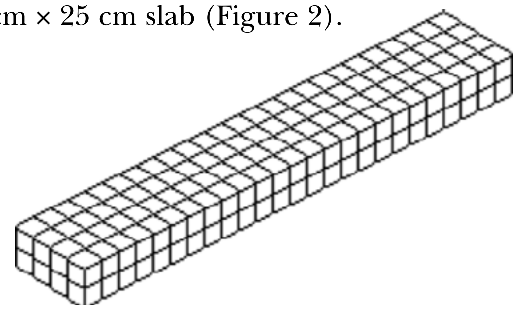

Figure 2. A 2 cm x 4 cm x 25 cm slab.

Also consider the 4 cm × 5 cm × 10 cm brick (Figure 3).

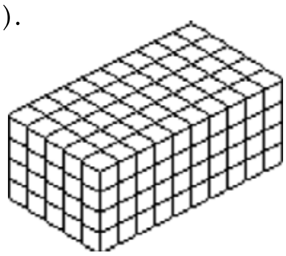

Figure 3. A 4 cm x 5 cm x 10 cm brick.

And consider the 5 cm × 5 cm × 8 cm brick (Figure 4).

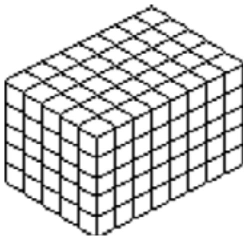

Figure 4. A 5 cm x 5 cm x 8 cm brick.

For each different 200 ml brick, measure the surface area.

Measure the total length of the edges of the brick.

(Adventurous students might try finding a cube, or a shape that is almost a cube, with a volume of 200 ml.)

Extension: Use 200 centicubes to make a step-pyramid. The first three steps look like this (Figure 5).

Figure 5. A three-step pyramid.

How many cubes are used by the third step? What is the surface area?

Explore the patterns of volume and surface area with further steps.

Activity 3 — Volume versus capacity: Context determines meaning and interpretation of technical vocabulary

The two mathematical words “volume” and “capacity” are easily confused. Strictly, “volume” refers to an amount of three-dimensional space. In contrast to this “capacity” specifically refers to an ability of a three-dimensional object to contain or hold some other amount of three-dimensional space. The tricky thing is that the object that does the containing has its own physical volume. That is, because the container is made with physical material, the space occupied by that material is the mathematical volume of the container, while the amount of space the container can contain is its capacity. Moreover, if we try to explain “capacity” as a measure of empty space, we need to realise that unless we have in mind a perfect vacuum, the “empty” space will be full of air (although in this case the word “full” seems to defy common sense), or have other particles of some kind in it. The distinction between the material of the container, and the space it contains, is less obvious when the materials that make the container are thin, or, in the case of clear glass or plastic, transparent. However, exploring thick-walled opaque containers helps clarify the two words.

Use centicube materials to make a container with outer dimensions 7 cm × 7 cm × 9 cm, that is, four 6 cm × 8 cm walls, each 1 centimetre thick, with a one-centimetre-thick 7 cm × 7 cm floor (Figure 6). This lidless box “contains” a 200 ml volume.

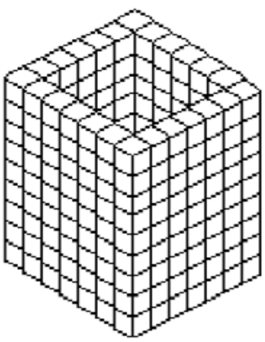

Figure 6. A lidless 7 cm x 7 cm x 9 cm box that “contains” a 200 ml volume.

What is the volume of the centicubes that make this container?

Can you explain the difference between the terms “volume” and “capacity”?

How are these two technical terms handled: in your local curriculum; in your school’s curriculum; across year levels; in your textbooks and worksheets?

What is the volume of the physical material that makes a 200 ml kitchen measuring cup? What about an inner volume of 200 ml with dimensions 10 cm × 10 cm × 2cm: what is the material volume of a centimetre-thick container for this configuration?

Consider an inner volume of 4 cm × 5 cm × 10 cm. What are the whole-number factors of 200? What are the surface areas of the contained volume and the containing cup?

Activity 4 — A gallery of clay cups: A constant volume of container can have different capacities

In this activity use pottery clay or an air-drying equivalent for permanent work, or use or playdough or plasticine for temporary work. Having different colours will make the results more visually interesting.

Each student is given a carefully measured 200 ml volume of dough. This is used to make a cup, or jug, with or without a handle, that uses all of the 200 ml ration of dough.

Students can sort their completed cups (or jugs) by height, width, thickness, shape (e.g., open box, open prism, open cylinder, part-cone, hemisphere, etc.) and then measure how much each cup (or jug) can contain. Adventurous students might then explore the mathematical challenges of a honey bee’s comb: maximum inner containment for honey, with minimum use of wax wall-material, along with issues of two-dimensional tessellation of cross-sections of hive-cells of different two-dimensional shapes.

Activity 5 — Volume and shape versus mass: A dense argument over misconceptions

Using stiff cardboard make several 200 ml rectangular-block containers, some with the same sizes for length and width and height, and some with different sizes (ignore the thickness of the cardboard, but aim to make the boxes have overall total (inner) volume as close as possible to 200 ml).

Devising two-dimensional "nets" for these rectangular boxes can be a challenge.

Find the (external surface) area of the nets (ignoring any flaps used to join faces, or walls, when making the completed boxes).

Measure the total length of the edges of the completed boxes.

Find different material to fill these rectangular boxes. Consider, for example:

- sand;
- rice;
- polystyrene (either blocks, cut to size, or packing beads);
- nails;
- paper, cut to fit flat, sheet by sheet;
- polyurethane paint (e.g., Estapol) that dries as a rigid plastic;
- matchsticks, balsa wood, or sawdust; or
- plaster of Paris, or Polyfilla (which dries solid without shrinkage).

It is easy to think that a small object will usually be lighter than a larger object, and vice versa. It is similarly easy to think that objects that are the same size will have the same mass. On the other hand, if objects have different shapes we often think they have different volumes, and probably different masses.

Only the challenge of feeling and measuring a variety of objects, with different sizes, different shapes, and different masses can alert us to the way appearances can be deceiving. Similarly, only the experience of seeing a collection of shapes that all look the same but have different masses can trigger heightened awareness of density of matter as a measurable variable. This is especially so when we lift up one block, then another, and they feel different.

(Note: the preceding paragraphs unthinkingly mixed words such as "light" and "mass". You might investigate the difference between "weight" and "mass", and consider young students' ability to grasp subtle concepts of physics, such as the weaker gravitational attraction on the Moon's surface, compared with that on Earth.)

Activity 6 — Squares and almost-squares: When whole-numbers do not yield an answer

A one-centimetre-thick square "slab" shape made using centicubes, with sides 14 cm × 14 cm, has a volume slightly less than 200 cubic centimetres. How many centicubes need to be added to make 200 cubic centimetres?

A larger 1 cm thick square slab of centicubes, with sides 15 cm × 15 cm is considerably more than 200 cubic centimetres. How much do you need to take from the 15 cm × 15 cm × 1 cm slab so that its volume is 200 cubic centimetres? Estimate the dimensions of a 1 cm thick square with a volume of 200 cubic centimetres. You can make an approximation of this, using 200 ml of dough, carefully shaped into a square that is 1 cm thick.

How long is the side of the square?

This can be an entry point to the concept of square roots.

(Note that this is a square-based version of the challenge task, mentioned in Activity 2, of making a cube that has a volume of 200 ml.)

Activity 7 — Kitchen cup survey

Part 1: Our diverse home cups

Encourage students to bring from home their family's kitchen measuring cup, along with other commonly used cups.

Check them all for their capacity and measure their dimensions and mass.

Sort the cups according to shape, material, size, etc.

The actual measuring cups are checked for accuracy of their graduated scale, along with checking for variations: cl, dl, ml, cup-fractions, and other non-metric alternatives.

Part 2: Cup-based recipes from home

Make a survey/search of simple recipes that use cups.

Where feasible, the simpler recipes can actually be made or cooked in the classroom (or at home). This can lead to questions about the final volume and mass of items that have been cooked, such as muffins, biscuits, breads, and cakes. Compare the mixing bowl full of mixture (chocolate crackles, fudge, biscuits, scones, etc.) with the tray-full of completed (cooked) edibles — compact volume versus dispersed volume.

Conclusions

This article is, fittingly, a little like something in a recipe book: suggestions for whetting the mathematical appetite. Like recipes, the real value of the article comes from actually using it, or parts of it, and from the responses of those who "taste" it. I expect that students (and teachers) who make these unusual cups and work with these activities will frequently stop and say something such as, "That's odd," or "That's amazing," or 'I didn't expect that!" Watch for those moments: they signal the start of fresh learning and reflective thinking.

References

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